

2020

MATHEMATICS — HONOURS

Seventh Paper

(Module - XIII)

Full Marks : 50

The figures in the margin indicate full marks.

*Candidates are required to give their answers in their own words
as far as practicable.*

\mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} respectively denote the set of natural numbers, integers, rational, real numbers and complex numbers.

Group - A

[Analysis - IV]

(Marks : 20)

Answer *any one* question.

1. (a) Let f and ϕ be two functions of x such that for some positive real number λ , $0 < f(x) \leq \lambda\phi(x)$ for all $x \geq a$. If each of f and ϕ be integrable on $[a, X]$ for every $X > a$, prove that $\int_a^\infty f(x)dx$ converges

if $\int_a^\infty \phi(x)dx$ converges and $\int_a^\infty \phi(x)dx$ diverges if $\int_a^\infty f(x)dx$ diverges.

- (b) Test the convergence of the integral $\int_0^1 \frac{\sqrt{x}}{e^{\sin x} - 1} dx$.

- (c) Establish the convergence of $\int_0^\infty \frac{x \log x}{(1+x^2)^2} dx$ and find its value. 8+6+6

2. (a) Show that $\int_0^1 \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}} dx$ is convergent.

Please Turn Over

(b) State Abel's Test in connection with the convergence of improper integral of product of two functions

over a bounded and closed interval. Using it, show that $\int_0^1 \frac{\log_e(1+x) \sin \frac{1}{x}}{x} dx$ is convergent.

(c) Express $\int_0^1 x^m (1-x^p)^n dx$ in terms of Beta function mentioning the conditions on m, n, p . Hence

evaluate $\int_0^1 x^5 (1-x^3)^{10} dx$. 6+(2+4)+8

3. (a) Show that for $f(x) = \cos kx$ on $[-\pi, \pi]$, where k is not an integer,

$$\cos kx = \frac{\sin kx}{\pi} \left[\frac{1}{k} - \frac{2k \cos x}{k^2 - 1^2} + \frac{2k \cos 2x}{k^2 - 2^2} + \dots \right].$$

Deduce that $\pi \cos k\pi = \frac{1}{k} + 2k \sum_{n \in \mathbb{N}} \frac{1}{k^2 - n^2}$.

(b) Evaluate $\int_0^1 dy \int_y^1 e^{x^2} dx$. 12+8

4. (a) Show that the integral $\iint_E e^{\frac{y-x}{y+x}} dx dy$, where E is the triangle with vertices at $(0, 0)$, $(0, 1)$ and

$$(1, 0) \text{ is } \frac{1}{4} \left(e - \frac{1}{e} \right). \quad 10$$

Or,

Evaluate $\iint_E x^{1/2} y^{1/3} (1-x-y)^{2/3} dx dy$, where E is the region bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$. 10

(b) Show that the volume included between the elliptical paraboloid $2z = \frac{x^2}{p} + \frac{y^2}{q}$, the cylinder

$$x^2 + y^2 = a^2 \text{ and the } xy \text{ plane is } \frac{\pi a^4 (p+q)}{8pq}. \quad 10$$

(3)

P(III)-Mathematics-H-7(Mod.-XIII)

Or,

Let a function f be defined on a rectangle $R = [0, 1; 0, 1]$ as follows :

$$f(x, y) = \begin{cases} \frac{1}{2} & \text{when } y \text{ is rational} \\ x & \text{when } y \text{ is irrational} \end{cases}$$

Show that (i) $\int_0^1 dy \int_0^1 f(x, y) dx = \frac{1}{2}$ and (ii) $\int_0^1 dx \int_0^1 f(x, y) dy$ does not exist. 4+6

Group - B

[Metric space]

(Marks : 15)

5. Answer **any one** question :

- (a) (i) For any two distinct points a, b in a metric space (X, d) , prove that there exist disjoint open spheres with centres at a and b respectively.
- (ii) In the metric space of real numbers (\mathbb{R}, d) with the usual metric, let $\rho(A, B)$ be the distance between two subsets A, B of \mathbb{R} . Show that $\rho(A, B) = 0$ where $A = \mathbb{N}$ and $B = \left\{ n + \frac{1}{2n} : n \in \mathbb{N} \right\}$. 15
- (b) (i) Let (X, d) be a metric space and $A, B \subset X$. Then show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ (\overline{U} denote the closure of U).
- (ii) If $\delta(A)$ and \overline{A} denote diameter and closure of a set A in a metric space (X, d) , then prove that $\delta(A) = \delta(\overline{A})$. 15
- (c) (i) Consider the metric space (\mathbb{R}^2, d) where $d(x, y) = |x_1 - y_1| + |x_2 - y_2|$ for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in \mathbb{R}^2$ for $a = (0, 0) \in \mathbb{R}^2$ and any positive number r , describe the open ball $S(a, r)$ geometrically.
- (ii) Let $C[a, b]$ be the set of all real valued continuous functions defined on $[a, b]$. Define $d(f, g) = \sup_{f, g \in C[a, b]} |f(t) - g(t)|$. Show that $A = \left\{ f \in C[a, b] : \inf_{x \in [a, b]} f(x) > 0 \right\}$ is an open set. 15
- (d) Prove that $C[a, b]$, the set of all real valued continuous functions defined on $[a, b]$, is complete under the metric d where $d(f, g) = \sup \{ |f(x) - g(x)| : a \leq x \leq b \}$ for all $f, g \in C[a, b]$. 15

Please Turn Over

- (e) (i) If a sequence $\{x_n\}_n$ is convergent in the metric space (X, d) , then prove that for $a \in X$, the set $\{d(x_n, a) : n \in \mathbb{N}\}$ is bounded.
- (ii) In the metric space (\mathbb{R}, d) with usual metric, consider the sequence $\{F_n\}_n$ of sets where

$$F_n = \left[-5 - \frac{1}{n}, -5\right] \cup \left[5, 5 + \frac{1}{n}\right] \text{ for all } n \in \mathbb{N}. \text{ Show that } \bigcap_{n=1}^{\infty} F_n \text{ is not singleton.} \quad 15$$

Group - C

[Complex Analysis]

(Marks : 15)

6. Answer *any two* questions :

- (a) (i) Show that the image of a line T under the stereographic projection is a circle minus north pole

in the Riemann sphere $x^2 + y^2 + \left(z - \frac{1}{2}\right)^2 = \frac{1}{4}$.

- (ii) Show that the function $\frac{\bar{z}}{z}$ is not continuous at the origin $Z=0$ for any choice of $f(0)$. 5+2½

- (b) (i) If $f(z)$ and $\overline{f(z)}$ are both analytic in a region, then show that $f(z)$ is constant in that region.

- (ii) Prove or disprove : If $f : S \rightarrow \mathbb{C}$ is differentiable on S , where $S \subseteq \mathbb{C}$ and $f'(z) = 0$ for all $z \in S'$, then f is a constant function on S . 5+2½

- (c) Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be defined by

$$f(z) = \begin{cases} \frac{(\bar{z})^2}{z}, & \text{for } z \neq 0 \\ 0 & , \text{ for } z = 0 \end{cases}$$

Show that Cauchy-Riemann equations are satisfied at $z = 0$, but the derivative of f fails to exist there. 7½

- (d) If $f(z)$ is an analytic function of $z = x + iy$, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |\operatorname{Re} f(z)|^2 = 2|f'(z)|^2$. 7½

- (e) Use Milne-Thompson method to find an analytic function whose imaginary part is given by :

$$v(x, y) = 3x^2y + y^3. \quad 7½$$
