2020

STATISTICS — HONOURS

Paper : CC-5

(Linear Algebra)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer ***any ten*** questions : 1×10

(a) Express $A = \begin{bmatrix} 4 & 5 & 1 \\ -2 & 7 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

(b) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, find A^{50}.

(c) Find the value of the determinant $\begin{bmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{bmatrix}$.

(d) If A and B are matrices of order n such that $A + B = I_n$ and $AB = 0$, show that A and B are both idempotent.

(e) If $AB = B$ and $BA = A$, show that both the matrices A and B are idempotent.

(f) For a real orthogonal matrix A, show that $|A^{-1}| = |A|$.

(g) If A is a real skew symmetric matrix and $(I + A)$ is non-singular, show that $(I + A)^{-1}(I - A)$ is orthogonal.

(h) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, find the rank of $A + A^2$.

(i) If A is a real non-symmetric matrix of order 3, find the determinant of the matrix $A - A^T$.

(j) If $S = \{(x, y, 0) : x, y \in \mathbb{R}\}$ and $T = \{(0, y, z) : y, z \in \mathbb{R}\}$, find $\dim(S \cap T)$.

(k) If $S = \{(x, 0) : x \in \mathbb{R}\}$ and $T = \{(u, u) : u \in \mathbb{R}\}$, then show that $\mathbb{R}^2 = S + T$, Here '+' denotes a sum of two vector sub spaces.

(l) If $S = \{(x, 0) : x \in \mathbb{R}\}$ and $T = \{(u, u) : u \in \mathbb{R}\}$ modify T, so that $\mathbb{R}^2 = S \oplus T$

Here $\oplus$ denotes the direct sum of two vector sub spaces.

(m) Let λ be an eigenvalue of a non-singular matrix A and show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

(n) If A is a real positive definite matrix, then show that A^{-1} is also positive definite.

2. Answer ***any four*** questions : 5×4

(a) If $S_r = \alpha^r + \beta^r + \gamma^r$, prove that $\begin{vmatrix} s_0 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix} = (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2$.

(b) If A is a non-singular matrix such that sum of each row (column) is k, then prove that sum of each row (column) of A^{-1} is $1/k$.

(c) "If A and B are real orthogonal matrices of same order and $|A| + |B| = 0$, then $(A + B)$ is a singular matrix."— Prove or disprove.

(d) What do you mean by the null space $\mathcal{N}(B)$ of an $m \times n$ matrix B? Show that

$$\dim(\mathcal{N}(B)) = n - \text{rank}(B).$$

(e) What is rank factorisation of a matrix? State and prove the rank factorisation theorem.

(f) The following are four vectors spanning a sub space $S : \{\underline{\alpha}_1, \underline{\alpha}_2, \underline{\alpha}_3, \underline{\alpha}_4\}$ where

$\underline{\alpha}_1 = (1, 2, 3, -1)'$, $\underline{\alpha}_2 = (2, 2, 6, -1)'$, $\underline{\alpha}_3 = (2, 2, 6, -1)'$ and $\underline{\alpha}_4 = (3, 3, 7, 0)'$. Find the dimension of S and a basis of $S^{\perp}$.

3. Answer ***any two*** questions : 10×2

(a) Find the determinant of the following matrix :

(i) $$\begin{vmatrix} 1 & \rho & \rho^2 & \rho^3 & \cdots & \rho^{n-1} \\ \rho & 1 & \rho & \rho^2 & \cdots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \rho & \cdots & \rho^{n-3} \\ . & . & . & . & \cdots & . \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \rho^{n-4} & \cdots & 1 \end{vmatrix}$$ where $|\rho| < 1$.

(ii) Let A be a non-singular matrix of order n and $\underline{U}$ and $\underline{V}$ be two vectors of order n. Then, show that $\left(A+UV'\right)$ is invertible if and only if $V'A^{-1}U \neq 1$ and in that case

$$\left(A+UV'\right)^{-1} = A^{-1} - \frac{A^{-1}UV'A^{-1}}{1+V'A^{-1}U}.$$

(iii) Use the above result to find the inverse of the matrix $M = (a-b)I_n + b\,1_n 1_n'$ where $a \neq b$ and $a+(n-1)\,b \neq 0$.

(b) (i) Derive Gram–Schmidt process of orthogonalisation of a given set of independent vectors.

(ii) Show that the following set of vectors in $\mathbb{R}^3$ is an independent set and orthogonalize them in Gram-Schmidt process to get an orthogonal basis of $\mathbb{R}^3$ —

$$\left\{(1,2,1)';(2,1,2)';(1,0,-1)'\right\}.$$

(c) (i) If A is a real symmetric matrix of order n, then show that there exists an orthogonal matrix P such that $PAP' = \Lambda$ and $PP' = I$,

where $\Lambda = \text{Diag}(\lambda_1, \lambda_2, ..., \lambda_n); \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$, being the distinct eigenvalues.

(ii) Discuss how the above result may be used to find the higher powers of A and also the inverse of A, when A is non-singular.

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