2021

MATHEMATICS — HONOURS

Paper : CC-1

(Unit : 1, 2, 3)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all the following multiple choice questions. Each question has four possible answers, of which exactly one is correct. Choose the correct option and justify your answer. (1+1)×10

(a) There is a point of inflexion of the curve $y = c\sin\left(\frac{x}{a}\right)$ where it meets

(i) the x-axis (ii) the y-axis

(iii) both the x-axis and the y-axis (iv) none of these.

(b) The horizontal asymptote of $f(x) = \frac{x-2}{2x+1}$ is

(i) $y = -2$ (ii) $y = 0$

(iii) $y = 1/2$ (iv) $y = -\frac{1}{2}$.

(c) The envelope of family of curve $y = mx - 2am - am^3$, m being parameter is

(i) $27ay^2 = 4(x + 2a)^3$ (ii) $4ay^2 = 27(x + 2a)^3$

(iii) $27ay^2 = 4(x - 2a)^3$ (iv) $4ay^2 = 27(x - 2a)^3$.

(d) Arc length of curve $y = x^{3/2}$, from (0, 0) to (4, 8) is

(i) $\frac{8}{27}\left(10^{2/3} + 1\right)$ (ii) $\frac{8}{27}\left(10^{3/2} - 2\right)$

(iii) $\frac{8}{27}\left(10^{3/2} - 1\right)$ (iv) $\frac{8}{27}\left(10^{3/2} + 1\right)$

(e) The polar equation of the tangent at α to a parabola with the latus rectum $4a$ can be expressed in the form

(i) $r^2 = a^2\sec^2\theta$ (ii) $r = a\sec\frac{\alpha}{2}\sec\left(\theta - \frac{\alpha}{2}\right)$

(iii) $r = a^2\sec^2\frac{\alpha}{2}\sec\left(\theta - \frac{\alpha}{2}\right)$ (iv) none of these.

(f) The foot of perpendicular drawn from origin to plane is (1, 2, 3). The equation of the plane is

(i) $x - 2y + 3z = 0$ (ii) $x + 2y + 3z = 0$

(iii) $x + 2y + 3z = 14$ (iv) $x - 2y - 3z = 14$.

(g) The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar if

(i) $k = 0$ or -1 (ii) $k = 1$ or -1

(iii) $k = 0$ or -3 (iv) $k = 3$ or -3.

(h) Coordinates of the points where the line $\frac{x}{2} = \frac{y}{1} = \frac{z}{3}$ intersects the sphere $x^2 + y^2 + z^2 = 56$ are

(i) $(4, -2, 6)$ (ii) $(-4, -2, -6)$

(iii) $(-4, -2, 6)$ (iv) $(4, -2, 6)$ and $(-4, -2, -6)$.

(i) $\left(\vec{b} \times \vec{c}\right) \cdot [(\vec{c} \times \vec{a}) \times (\vec{a} \times \vec{b})] =$

(i) $(\vec{a} \times \vec{b}) \cdot \vec{c}$ (ii) $\left\{(\vec{b} \times \vec{a}) \cdot \vec{c}\right\}^2$

(iii) $\left\{(\vec{a} \times \vec{b}) \cdot \vec{c}\right\}^2$ (iv) $\vec{a} \cdot (\vec{b} \times \vec{c})$

(j) If $\vec{a} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ and $\vec{b} = 2t\hat{i} + \hat{j} - t\hat{k}$, then at $t = 0, \frac{d}{dt}\left(\vec{a} \times \vec{b}\right) =$

(i) $2\hat{i} + 2\hat{j}$ (ii) $-2\hat{i} + \hat{j}$

(iii) $-\hat{i} + 2\hat{j}$ (iv) $-2\hat{i} + 2\hat{j}$

2. Answer ***any three*** questions :

(a) (i) Evaluate : $\lim_{x \to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(ii) Evaluate : $\lim_{x \to \infty}\left[\sqrt[3]{(a+x)(b+x)(c+x) - x}\right]$, where a, b, c are positive constants. 3+2

(b) If $f(x) = \tan x$, prove that $f^n(0) - {}^nc_2 f^{n-2}(0) + {}^nc_4 f^{n-4}(0)\ldots = \sin\frac{n\pi}{2}$. 5

(c) Prove that the asymptotes of the cubic $(x^2 - y^2)y - 2ay^2 + 6x - 9 = 0$ form a triangle of area a^2. 3+2

(d) If $I_n = \int_0^{\pi/2} x\sin^n x\,dx, n > 1$, show that $I_n = \frac{n-1}{n}I_{n-2} + \frac{1}{n^2}$. Hence evaluate $\int_0^{\pi/2} x\sin^5 x dx$. 3+2

(e) Find the area of the loop of the curve $xy^2 + (x + a)^2(x + 2a) = 0,\ a > 0$. 5

3. Answer ***any four*** questions : 5×4

(a) PSP' is a focal chord of the conic $\frac{l}{r} = 1 + e\cos\theta$. Prove that the angle between the tangents at P and P' is $\tan^{-1}\frac{2e\sin\alpha}{1-e^2}$, where α is the angle between the chord and the major axis.

(b) The normals at the ends of the latus rectum of the parabola $y^2 = 4ax$ meet the curve again in Q and Q'. Prove that $QQ' = 12a$.

(c) Find the length and the equation of the line of shortest distance between the lines

$3x - 9y + 5z = 0 = x + y - z$ and $6x + 8y + 3z - 13 = 0 = x + 2y + z - 3$.

(d) Show that the equation of the plane through the intersection of the planes $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ and perpendicular to the xy plane is

$$(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0$$

(e) If the lines $x = ay + b = cz + d$ and $x = \alpha y + \beta = \gamma z + \delta$ are coplanar, then show that $(\gamma - c)(a\beta - b\alpha) = (\alpha - a)(c\delta - d\gamma)$.

(f) A variable sphere passes through the origin O and meets the axes in A, B, C so that the volume of the tetrahedron $OABC$ is constant. Find the locus of the centre of the sphere.

(g) Find the equations of the generating lines of the hyperboloid $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{16} = 1$ passing through the point $\left(2, -1, 4/3\right)$.

4. Answer ***any two*** questions :

(a) Show that if the straight lines $\vec{r} = \vec{a} + u\vec{\alpha}$ and $\vec{r} = \vec{b} + v\vec{\beta}$ intersect, then $\left(\vec{a} - \vec{b}\right)\cdot\vec{\alpha}\times\vec{\beta} = 0$ but $\vec{\alpha}\times\vec{\beta} \neq \vec{0}$. 5

(b) The line of action of the force $\vec{f}=(1,-1,2)$ passes through the point $A(2, 4, -1)$. Find its moment about an axis through the point $P(3, -1, 2)$ and having the direction $2\hat{i}-\hat{j}+2\hat{k}$. 5

(c) (i) If $\vec{\alpha}=t^2\hat{i}-t\hat{j}+(2t+1)\hat{k}$ and $\vec{\beta}=(2t-3)\hat{i}+\hat{j}-t\hat{k}$, then find $\frac{d}{dt}\left(\vec{\alpha}\times\frac{d\vec{\beta}}{dt}\right)$ at $t=2$.

(ii) If $\vec{r}(t)=2\hat{i}-\hat{j}+2\hat{k}$ when $t=2$ and $\vec{r}(t)=4\hat{i}-2\hat{j}+3\hat{k}$ when $t=3$, then evaluate $\int_2^3\left(\vec{r}\cdot\frac{d\vec{r}}{dt}\right)dt$.

3+2
