# 2021

## MATHEMATICS — HONOURS

**Paper : DSE-A(2)-2**

**(Mathematical Modelling)**

**Full Marks : 65**

*The figures in the margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

### Group - A

**1.** Answer ***all*** questions : 2×10

(To each question there are four alternatives of which one is correct. Choose the correct one with proper justification. One mark is for correct answer and one mark is for proper justification.)

(a) Laplace transform of a function $f(t)$ is given by

(i) $F(s)=\int_0^\infty f(t)e^{st}dt$, where $s>0$ (ii) $F(s)=\int_0^\infty f(t)e^{-st}dt$, where $s>0$

(iii) $F(s)=\int_0^\infty f(s)e^{-st}dt$, where $s>0$ (iv) $F(s)=\int_0^\infty f(-s)e^{-st}dt$, where $s>0$.

(b) The point $x = 0$, in the standard Bessel's equation is a (an)

(i) ordinary point (ii) regular singular point

(iii) irregular singular point (iv) none of these.

(c) Legendre polynomial $P_1(x)$ is given by

(i) 1 (ii) $x/2$

(iii) $x$ (iv) $x^2$.

(d) Using linear congruence method $x_{n+1} = 7x_n + 13 \bmod (97)$ with seed $x_0 = 105$, three random numbers generated are respectively

(i) 63, 13 and 90 (ii) 13, 34 and 78

(iii) 69, 11 and 90 (iv) none of these.

(e) Number of extreme points of the convex set $x^2 + y^2 \leq 1$ are

(i) two (ii) three

(iii) countably infinite (iv) uncountably infinite.

(f) If $Pn(x)$ is Legendre polynomial of degree $n$, then

(i) $P_n(-x) = P_n(x)$ (ii) $P_n(-x) = (-1)^n P_n(x)$

(iii) $P_n(-1) = 1$ (iv) none of these.

(g) Cars arrive at an automated car wash following Poisson distribution. If their arrival rate is 20 per hour, and it takes exactly 2 minutes for a car wash, what is the average waiting time in line?

(i) 1 minute (ii) 2 minutes

(iii) 4 minutes (iv) 3 minutes.

(h) The pattern in which the customer change his queue in the initial time and finally join the shortest queue is

(i) Balking (ii) Reneging

(iii) Jockeying (iv) Collusion.

(i) The set of regular singular points of the ordinary differential equation

$$x\left(x^3+x^2-x-1\right)^2\frac{d^2y}{dx^2}-\left(x^4+3x^3+3x^2+x\right)\frac{dy}{dx}+\left(x^4-5x^2+4\right)y=0 \text{ is}$$

(i) $\{0, -1\}$ (ii) $\{0\}$

(iii) $\{0, 1, -1\}$ (iv) none of these.

(j) The zeros of Legendre polynomial of 1st kind can be

(i) any real number

(ii) any real or purely imaginary number

(iii) any real numbers between $-1$ and 1

(iv) only positive proper fraction.

## Group - B

### Unit - 1

**2.** Answer ***any two*** questions :

(a) (i) $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-4^2\right)y=0$ near $x=0$.

(ii) Solve the initial-value problem $\frac{d^2y}{dt^2}-\frac{dy}{dt}-2y=18e^{-t}\sin 3t,\ y(0)=0,\ y'(0)=3,$ using Laplace transform method. 5+5

(b) (i) Prove the Recurrence Formulae for Legendre Polynomials

$$(2n+1)\times P_n(x) = (n+1)\ P_{n+1}(x) + n\ P_{n-1}(x).$$

(ii) Using generating function for $P_n(x)$, prove the following :

$$P_0(x)=1,\ P_1(x)=x,\ P_2(x)=\tfrac{1}{2}\left(3x^2-1\right),\ P_3(x)=\tfrac{1}{2}\left(5x^3-3x\right) \text{ and } P_4(x)=\tfrac{1}{8}\left(35x^4-30x^2+3\right).$$

5+5

(c) (i) Show that $L\left\{e^{-2t}(3\cos 6t-5\sin 6t)\right\}=\dfrac{3(p-8)}{p^2+4p+40}$.

(ii) Evaluate $L^{-1}\left(\dfrac{2s+1}{(s-1)^2(s+2)^2}\right)$. 5+5

(d) (i) If $J_n(x)$ is Bessel function of 1st kind, then show that $xJ_n'(x)=n\,J_n(x)-x\,J_{n+1}(x)$ and hence deduce that $\dfrac{d}{dx}\left(x^{-n}J_n(x)\right)=-x^{-n}\,J_{n+1}(x)$.

[Here Prime ( $'$ ) denotes the 1st order derivative with respect to $x$.]

(ii) Using Rodrigue's formula, prove that $P_{n+1}'(x)-P_{n-1}'(x)=(2n+1)P_n(x)$ where $P_n(x)$ is the Legendre polynomial of 1st kind of degree $n$.

[Here Prime ( $'$ ) denotes the 1st order derivative with respect to $x$.] 5+5

## Unit - 2

**3.** Answer ***any five*** questions :

(a) Write a short note on Monte Carlo simulation method to find the area of the bounded region

$$x^2+y^2\le 25,\ x\ge 0,\ y\ge 0.$$ 5

(b) Describe the middle-square method to generate random numbers starting with a four-digit positive integer (seed, say $x_0$) and state one of its major drawback. 4+1

(c) A small harbour has unloading facility for ships. Only one ship can be unloaded at any one time. Ships arrive for unloading of cargo at the harbour. Time-span between the arrival of successive ships and unloading time of the five ships are given in the table below. Find the average time spent by the ships in the harbour. 5

| | Ship 1 | Ship 2 | Ship 3 | Ship 4 | Ship 5 |
|---|---|---|---|---|---|
| Time between successive ships (in minutes) | 15 | 35 | 20 | 100 | 35 |
| Unloading time (in minutes) | 65 | 55 | 60 | 70 | 80 |

(d) A firm makes two products $P_1$ and $P_2$ and has production capacity of 18 tonnes per day. $P_1$ and $P_2$ require same production capacity. The firm must supply at least 4t of $P_1$ and 6t of $P_2$ per day. Each tonne of $P_1$ and $P_2$ requires 60 hours of machine work each. Maximum machine hours available are 720. Profit per tonne for $P_1$ is Rs. 160 and $P_2$ is Rs. 240. Find optimal solution by graphical method. 5

(e) Given an average arrival of 20 per hour, is it better for a customer to get a service at a single channel with mean service rate of 22 customers or at one of the two channels in parallel, with the mean service rate of 11 customers for each of the two channel? Assume that both queues are (M/M/s: $\alpha$/FIFO). 5

(f) Solve the Problem Algebraically to get an Optimal Solution : 5

Maximize $25x_1 + 30x_2$

Subject to

$$20x_1 + 30x_2 \leq 690$$
$$5x_1 + 4x_2 \leq 120$$
$$x_1, x_2 \geq _0 \geq 0$$

(g) Following is the optimal table of the LPP

Maximize $Z = -x_1 + 2x_2 - x_3$

Subject to
$$3x_1 + x_2 - x_3 \leq 10$$
$$-x_1 + 4x_2 + x_3 \geq 6$$
$$x_2 + x_3 \leq 4$$
and $x_j \geq 0, \quad j = 1, 2, 3.$

| $C_j \rightarrow$ | | | $-1$ | $2$ | $-1$ | $0$ | $0$ | $0$ | $-M$ |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $y_B$ | $X_B$ | $y_1$ | $y_2$ | $y_3$ | $y_4$ | $y_5$ | $y_6$ | $y_7$ |
| 0 | $y_4$ | 6 | 3 | 0 | $-2$ | 1 | 0 | $-1$ | 0 |
| 2 | $y_2$ | 4 | 0 | 1 | 1 | 0 | 0 | 1 | 0 |
| 0 | $y_5$ | 10 | 1 | 0 | 3 | 0 | 1 | 4 | $-1$ |
| $Z_j - C_j \rightarrow$ | | | 1 | 0 | 3 | 0 | 0 | 2 | $M$ |

(M is positive large number)

(i) Determine the range under which the component $C_2$ of the profit vector can vary so that the current optimal solution remains optimal.

(ii) Determine the range under which the component $b_2$ of the requirement vector can vary to maintain the optimality of the current optimal solution. 5

(h) A departmental store has only one cashier. During the rush hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be handled by the cashier is 24 per hour. Assuming the conditions for use of the single-channel queueing model, find out the followings :

(i) Probability that cashier is idle

(ii) Average number of customers in the system

(iii) Average time a customer in the system

(iv) Average number of customers in the queue

(v) Average time a customer spends in the queue. 1+1+1+1+1