2021

STATISTICS — HONOURS

Paper : CC-3

(Mathematical Analysis)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Throughout, m stands for 100 plus the number represented by the last two digits of your roll number; i.e. $m = 100 + 10x + y$ if the last digit of your roll number is y and the second last digit is x.

You should first put the value of m before solving each question. ***Do not*** work with general m and then replace it subsequently by the actual value.

1. Answer ***any ten***. If you answer more than ten, then only the first ten attempted will be checked. 2×10

(a) Decide if the set $\left\{\sin\frac{x}{m}+\frac{m}{x} : |x| \geq \frac{39}{m}\right\}$ is bounded above and below or not.

(b) Suppose $x_0 = 0$, $x_1 = \frac{1}{2}$ and for $n \geq 2$, $x_{n+1} = \frac{3}{2}x_n - \frac{1}{2}x_{n-1}$. Find x_{2m}.

(c) Find $\lim_{n\to\infty}\left(\sqrt[m]{m}-\frac{3}{n}\right)^{2m}$.

(d) Obtain $\lim_{x\to\infty}\frac{x^m+(\ln x)^m}{x^m \ln x}$, if it exists.

(e) Compute $\sum_{n=m}^{\infty}\frac{1}{(n+m)(n+m+1)}$.

(f) Decide if the series $\frac{1}{m} - \frac{2^m}{m^2} + \frac{3^m}{m^3} - \frac{4^m}{m^4} + \ldots + \frac{(-1)^n n^m}{m^n} + \ldots$ converges, and if so, whether absolutely.

(g) Give an example of two monotone functions f and g on $[0, m]$ such that $f \cdot g$ is not monotone.

(h) Obtain the radius of convergence of the power series $\sum_{n=m+1}^{\infty} n^m (mx)^n$.

Please Turn Over

(i) Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as $f(x) = \begin{cases} x(x-1)(x-2)...(x-m), & x \text{ rational}, \\ 0, & x \text{ irrational}. \end{cases}$

At which points in $\mathbb{R}$ is f continuous?

(j) For which values of the positive integer n is the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \frac{\sin^n x}{x^m}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

differentiable?

(k) Find a function $f : \mathbb{R} \to \mathbb{R}$ that has all derivatives at 2 and $f^{(n)}(2) = m^n$ for every $n \geq 1$.

(l) Evaluate $\int_0^{\frac{m}{2}} x^m (m-2x)\, dx$.

(m) Suppose $f : (0; \infty) \to \mathbb{R}$ is defined by

$$f(x) = \int_{\frac{x}{m}}^{mx} \frac{\sin u}{u^3}\, du.$$

Compute $f'(m)$.

(n) Compute the coefficient of x^m in expansion of $\sin mx + \cos mx$.

(o) Maximize $(x_1 x_2 \ldots x_m)^3$ subject to $x_1^2 + x_2^2 + \cdots + x_m^2 = m^3$.

2. Answer ***any three*** questions : 5×3

(a) Give an example of a sequence $(x_n)_{n \geq 1}$ such that the set of limits of subsequences of the sequence is the interval $\left[\frac{1}{m}, m\right]$.

(b) Let $f : [0, m] \to \mathbb{R}$ be Riemann integrable on $[0, m]$. Prove from the definition that the function $g : [0, m] \to \mathbb{R}$ defined by

$$g(x) = f(m - x)$$

is so too, with the same integral.

(c) Evaluate $\int_0^m \int_0^m \frac{dx\, dy}{(2m + mx + y)^2}$

(d) Show that the function $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(x) = m^x \cos x - \frac{1}{m}$$

has a root in $(-\pi, \pi)$.

(e) Find all stationary points of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = mx^2 + 3xy + my^2 + x - y - m$$

and determine for each whether it is a point of local minimum, local maximum or a saddle point.

3. Answer ***any three*** questions :

(a) (i) Prove that the sequence $((-1)^n m^{1/n})_{n \geq 1}$ does not converge.

(ii) Give a *complete* proof that there exists $x \in \mathbb{R}$ with $x^m = m$. 2+8

(b) If $(x_n)_{n \geq 1}$ is a sequence of positive real numbers such that for every $n \in \mathbb{N}$, $x_{n+m} < \frac{x_n}{m}$, show that the series $\sum_{n=1}^{\infty} x_n$ converges. 10

(c) If $f : [m, m + 1] \to \mathbb{R}$ is continuous on $[m, m + 1]$ and differentiable on $(m, m + 1)$ such that $\lim_{x \to m+} f'(x) = \frac{m}{2}$, show that f has right derivative $\frac{m}{2}$ at m. 10

(d) (i) Let a, b be real numbers with $a > 0$. If

$$f(x) = \begin{cases} \frac{e^{mx} - 1}{ax}, & x < 0, \\ b, & x = 0, \\ \frac{\ln \sin mx}{\ln x}, & x > 0, \end{cases}$$

then verify if a and b can be found to make f continuous.

(ii) If $f_n(x) = \dfrac{x^n}{m + x^n}$, $x \in (-1, 1]$ for $n \geq 1$, does (f_n) converge pointwise on $(-1, 1]$? Does it converge uniformly? Would your answers change if the domain is changed to $[-1, 1]$? Explain. 4+6

(e) (i) Suppose the sequence $(x_n)_{n \geq 1}$ converges to x in $\mathbb{R}$. Does the sequence

$$\frac{1}{n}\{x_{mn+1} + x_{mn+2} + \ldots + x_{2mn-1} + x_{2mn}\}, \quad n \geq 1,$$

always converge? If so, find its limit.

(ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and

$$f\left(\frac{n}{k\sqrt{m}}\right) = \frac{n^2}{k^2}$$

for every $n \in \mathbb{Z}$ and $k \in \mathbb{N}$. Show that $f(x) = mx^2 \ \forall x \in \mathbb{R}$. 3+7
