2021

STATISTICS — HONOURS

Paper : SEC-B-1

(Monte Carlo Method)

Full Marks : 80

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***any fifteen*** from question numbers **1-20**. 2×15

1. What is simulation? Why is it necessary?

2. Suppose you are given a coin and you want to find out the chance of landing it on edge. How do you use simulation to estimate the chance of landing on edge?

3. Consider the following random number generator :

$$x_{n+1} = (3x_n + 2)\ mod\ 7\ .$$

Generate first 4 numbers with $x_0 = 2$.

4. Consider the following random number generator

$$x_{n+1} = (11x_n + 13)\ mod\ 11\ .$$

Generate 4 uniform(–2,1) pseudorandom numbers using $x_0 = 5$.

5. Consider the random number generator :

$x_{n+1} = (5x_n + 1)\ mod 8\ .$

with $x_0 = 2$. After how many numbers, the seed x_0 will appear again?

6. If $\int_{-1}^{1} x^2 dx = E(g(X) \mid X \sim N(0,1),$ determine the function g.

7. Give a suitable Monte Carlo estimate of the integral $\int_0^\infty \exp(x)dx.$

8. Suppose you have access to 10000 random observations from Cauchy (0,1) distribution. How can you use these to estimate the integral = $\int_{-\infty}^{\infty} \frac{\exp(-x^2)}{1+x^2} dx$?

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9. Consider the random number generator

$$x_{n+1} = 11x_n \ mod\ 31$$

with $x_0 = 3$. Construct a scatterplot of the points $(x_{n-1},\ x_n), n = 1, 2, ..., 5$ and comment on the pattern.

10. "The shape of the histogram for 100000 random numbers from $\frac{1}{2}N(1,1) + \left(\frac{1}{2}\right)N(-1, 30)$ is expected to be bimodal" – Justify.

11. Describe a method of generating a single observation from Binomial (2, 1/3) distribution.

12. If you are given Uniform (0, 1) random numbers 0.1 and 0.23, generate two observations from a Bernoulli (0.11) distribution.

13. "Histogram of 10000 random numbers from Binomial (10, 0.5) will be bell-shaped with center at 0" – Justify.

14. If 10000 random observations are taken separately from Poisson (0.5) and Poisson (100) distributions and corresponding histograms are constructed, then both the histograms are positively skewed as Poisson distribution is a positively skewed distribution. Comment on this with justification.

15. What is a pseudo-random number? Why is it called "pseudo"?

16. Write down the Box-Muller transformation to generate a single observation from a standard normal distribution. Can you identify a drawback of this method?

17. Consider Monte Carlo estimation of the integral $\theta = \int_{-.05}^{.05} \left(\frac{1}{x}\right) dx$ based on *n iid* Uniform (– 0.05, 0.05) observations. Comment on the nature of the scatterplot of the points $(n,\ \hat{\theta}_n)$ with $\hat{\theta}_n$ as the Monte Carlo estimate of θ.

18. Suppose you are given 9 *iid* observations from a standard normal distribution. How do you generate two observations from a *t*-distribution with 8 degrees of freedom?

19. Describe a sample drawing scheme from a Beta (2, 2) distribution.

20. Consider the outcomes of tossing a fair coin 1000 times. What do you expect if a graph of $\frac{2X_n - n}{n}$ versus n for n = 1, 2, ..., 1000 is drawn, where X_n is the number of heads obtained in n tosses?

Answer ***any six*** from question numbers **21-28**. 5×6

21. Consider estimation of $\theta = P(1 < X < 2 \mid X \sim N(0,1)$ using Monte Carlo Method. Find the least sample size n ensuring at least 99% chance that the true value differs from the estimator by not more than 0.10.

22. Suppose you want to estimate $\theta = E_f(h(X)) = \int_{-\infty}^{\infty} h(x) f(x) dx$, with $f(x) = kf^*(x)$ for known f^* and unknown k. Show that an importance sampling estimator of $E_f(h(X)$ can be expressed as a weighted mean with suitable weights.

23. Suppose you want to estimate $\theta = P(X > 3 \mid X \sim N(0,1))$. Identify a potential drawback of Monte Carlo estimate for θ. How do you use importance sampling to get a reasonable estimate of θ?

24. Develop a scheme to obtain the Monte Carlo estimate of $\Gamma(1.5)$ based on n *iid* observations from Uniform (0, 1) distribution. If 4 observations from Uniform (0, 1) are 0.0078, 0.9352, 0.1080 and 0.0063, calculate your estimate.

25. Suppose you are given a biased six faced die, where the probability of obtaining any of the faces 1, 2, ..., 5 are equally likely and 6 appears with probability 0.10. Given 4 observations from Uniform (0, 1) as 0.18, 0.9352, 0.1080 and 0.0063, simulate 4 rolls of this die.

26. How do you use Monte Carlo Method to estimate the value of π based on n *iid* observations (x_i, y_i), where x_i and y_i are independent observations from Uniform (–1, 1)?

27. How do you use a full pack of 52 cards to generate random samples from a discrete uniform distribution over {1, 2, .., 13}? Justify your method with necessary derivations.

28. Suppose you toss two four-faced balanced die for simulating uniform random digits from 1 to 6 is as follows : throw the two dice and record the sum. Interpret 7 as 0 and ignore 8. Justify the procedure with necessary derivations.

Answer ***any two*** from question numbers **29-31**. 10×2

29. If U is a Uniform (0, 1) random variable then check with necessary calculations whether the following relations generate Uniform (0, 1) random numbers :

(i) 2Min(U, 1–U); (ii) 2Max(U, 1–U) and (iii) $(2U + 1) \pmod 1$.

30. Suppose *iid* observations are to be drawn from a truncated $N(0, 1)$ distribution over $(0, \infty)$. If you can generate *iid* observations from $N(0, 1)$ distribution, how do you get observations from the above truncated distribution? Justify your method with necessary mathematical derivations.

31. Suppose X is a random variable with the following pdf :

$$f(x) = x\, I(0 \le x \le 1) + (2 - x)\, I(1 \le x \le 2),$$

where $I(.)$ is the indicator function.

(i) Express $f(x)$ as $af_1(x)+(1–a)f_2(x)$, where $0 < a < 1$, $f_1(x)$ and $f_2(x)$ are pdfs.

(ii) Use the representation in (i) above, supported by proper mathematical justification, to generate a random observation from f.

(iii) Discuss any other scheme to generate a random observation from f.