# 2022

## MATHEMATICS — HONOURS

### Paper : CC-3

### (Real Analysis)

### Full Marks : 65

*The figures in the margin indicate full marks.*

*Candidates are required to give their answers in their own words as far as practicable.*

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$ denote the set of all natural, real and rational nos.

*Notations and symbols have their usual meanings.*

**1.** Answer *all* the following multiple choice questions. For each question 1 mark for choosing correct option and 1 mark for justification. 2×10

(a) Let $A = [5, 6)$ and $B = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$. Let $S = \{x - y : x \in A,\ y \in B\}$. Then $\operatorname{Inf} S$ is

(i) 3 (ii) 4

(iii) 5 (iv) 6.

(b) Let $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0, 2\}$ and $T = \bigcup_{n=1}^{\infty}\left(2 - \frac{1}{n+1}, 3 + \frac{1}{n}\right)$. Then $S \cap (\mathbb{R} - T)$ is

(i) open (ii) closed

(iii) both open and closed (iv) neither open nor closed.

(c) Let $A = \{[x] : 0 < x < 100\}$ and $B = \{2^i : i \in \mathbb{Z}\}$. Then $A \cup B$ is

(i) uncountable (ii) enumerable

(iii) finite (iv) empty.

($[x]$ denotes the largest integer not exceeding $x$)

(d) $\left\{\frac{5^n}{n!} + \left(\frac{3}{5}\right)^n\right\}$

(i) converges to 1 (ii) converges to 0

(iii) diverges to $+\infty$ (iv) converges to 3.

(e) Number of subsequential limits of $\left\{\dfrac{(-1)^{2n}\sin\dfrac{n\pi}{3}}{n^5}\right\}$ is

(i) 1 (ii) 2

(iii) 3 (iv) 5.

(f) Let $S$ be a bounded set of real numbers such that $S$ does not have a least element. Then

(i) Inf $S = -\infty$ (ii) each point of S is an isolated point

(iii) $S$ has at least one limit point (iv) $S$ fails to have any limit point.

(g) Let $S$ be a non-empty subset of $\mathbb{R}$, which of the following statement is true?

(i) If $x$ is a boundary pt. of $S$ then $x \in S$.

(ii) If $x$ is a limit pt. of $S$ then $x \in S$.

(iii) If $x$ is an isolated pt. of $S$ then $x \in S$.

(iv) If $x$ is an exterior point of $S$ then $x \in S$.

(h) Let $\{x_n\}_{n=1}^{\infty} = \{\sqrt{1}, -\sqrt{1}, \sqrt{2}, -\sqrt{2}, \sqrt{3}, -\sqrt{3}, ...\}$ and $z_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \forall n \in \mathbb{N}$. Then $\{z_n\}_{n=1}^{\infty}$ is

(i) unbounded above (ii) monotonic

(iii) bounded but not convergent (iv) convergent.

(i) Let $A = \left\{\frac{2}{z+1} : z \in (-1, 1)\right\}$. Then $A^d \setminus A$ is

(i) $\phi$ (ii) $(1, \infty)$

(iii) $\{1\}$ (iv) none of these.

(j) If $\{a_n\}_{n=1}^{\infty}$ is a monotone increasing sequence of real numbers and bounded above then the sequence $\left\{\dfrac{\sum_{i=1}^{n} a_i}{n}\right\}_{n=1}^{\infty}$ is

(i) bounded but not convergent (ii) always convergent

(iii) always divergent (iv) none of these.

## Unit - 1

Answer ***any four*** questions.

**2.** State Archimedean property of real numbers. Use it to show that between any two distinct real numbers there are infinitely many rational numbers. 1+4

**3.** Prove or disprove the following statements :

(a) If $S, T$ are non-empty bounded sets of real numbers and $V = \{xy : x \in S,\ y \in T\}$, then $\mathrm{Sup}V = \mathrm{Sup}S \times \mathrm{Sup}T$.

(b) The set $A = \{x \in \mathbb{R} : x + y \in \mathbb{Q}$ for some $y \in \mathbb{R}\}$ is countable. 3+2

**4.** (a) Show that union of two enumerable sets of real numbers is enumerable.

(b) Prove or disprove : If $S$ is a set of real numbers with its derived set consisting of exactly one point, then $S$ must be bounded. 3+2

**5.** (a) Prove or disprove : Every bounded infinite subset of $\mathbb{R}$ has an interior point.

(b) Let $a$ and $b$ be two irrationals such that $a < b$. Show that there is a rational number $q$ such that $a < q < b$. 2+3

**6.** (a) Define closed set. Give an example of a closed set which is non-empty and has no limit point in $\mathbb{R}$.

(b) Prove or disprove : $\mathbb{R} \setminus \{x \in \mathbb{R} : \sin x = 0\}$ is an open set. (1+1)+3

**7.** Prove that the derived set of any set in $\mathbb{R}$ is a closed set. Hence, show that $\{x \in \mathbb{R} : x^2 - 3x + 2 \leq 0\}$ is a closed set. 3+2

**8.** (a) Prove or disprove : The set $A$ of all open intervals with irrational end points is an uncountable set.

(b) Prove or disprove : Let $A$ and $B$ be any two subsets of $\mathbb{R}$. If $\inf(A) \subseteq \inf(B)$, $A^d \subseteq B^d$ and $\bar{A} \subseteq \bar{B}$ then $A \subseteq B$. 3+2

## Unit - 2

Answer ***any four*** questions.

**9.** '$l$' is a limit point of a set $S \subseteq \mathbb{R}$ if and only if there exists a sequence of distinct elements of $S$ converging to '$l$'. Establish this result. 5

**10.** Show that every monotonically increasing sequence which is bounded above is convergent.

Use this result to show that $\{x_n\}$ is convergent where $x_1 = \sqrt{13}$ and $x_n = \sqrt{13 + x_{n-1}}\ \forall n \geq 2$. 3+2

**11.** (a) Let $\{x_n\}$, $\{y_n\}$ be convergent sequence of real numbers such that $x_n \leq y_n\ \forall n \in \mathbb{N}$. Prove that $\lim\limits_{x\to\infty} x_n \leq \lim\limits_{x\to\infty} y_n$.

**Please Turn Over**

(b) Prove that $\left\{\frac{1}{n^2}+\frac{1}{(n+1)^2}+...+\frac{1}{(2n)^2}\right\}$ converges to zero. 2+3

**12.** (a) Let $\{a_n\}$ be a sequence of positive real numbers such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=l<1$. Show that $\lim_{n\to\infty}a_n=0$.

(b) Prove or disprove : If $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers such that $\{x_n y_n\}$ is convergent, then both $\{x_n\}$ and $\{y_n\}$ are bounded. 3+2

**13.** (a) Prove or disprove : A sequence of irrational numbers can not have a rational limit.

(b) Find the limit, if exists, of the sequence $\left\{\frac{x^n}{n!}\right\}_{n=1}^{\infty}$ where $x \in \mathbb{R}$. 3+2

**14.** State and prove Cauchy's general principle of convergence. 5

**15.** (a) Prove or disprove : Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers and $\lambda = Sup\{a_n : n \in \mathbb{N}\}$.

Then there is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ such that $\underset{k\to\infty}{Lt}\, a_{n_k}=\lambda$.

(b) If $|a_{n+1}-a_n|<\left(\frac{1}{2}\right)^n$ for all $n \in \mathbb{N}$, show that $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence. 3+2

## Unit - 3

Answer ***any one*** question.

**16.** State and prove Leibnitz test. Using it show that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ is convergent. 1+3+1

**17.** (a) State Cauchy's $n$-th root test. Use it to show that the series $\frac{1^3}{3}+\frac{2^3}{3^2}+1+\frac{4^3}{3^4}+...+\frac{n^3}{3^n}+...$ is convergent.

(b) Show that the series $\frac{1}{5}+\frac{1}{7}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{5^3}+\frac{1}{7^3}+...+\frac{1}{5^n}+\frac{1}{7^n}+....$ is convergent. (1+2)+2

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