2022

MATHEMATICS — HONOURS

Paper : CC-8

(Riemann Integration and Series of Functions)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ denote the sets of natural, real, rational numbers and integers respectively.

1. Answer the following multiple choice questions having only one correct option. Choose the correct option and justify your choice. $(1+1)\times 10$

(a) Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P, Q be any two partitions on $[a, b]$. Then

(i) $L(P \cup Q, f) \leq L(P, f)$ (ii) $U(P \cap Q, f) \leq U(Q, f)$

(iii) $L(P, f) \leq U(P \cup Q, f)$ (iv) $U(P, f) \leq U(P \cup Q, f)$.

(b) Identify the set which is not negligible.

(i) $\mathbb{Q}$

(ii) $\{x\sqrt{2} : x \in \mathbb{Z}\}$

(iii) The set of points of discontinuity of a monotone function on $\mathbb{R}$.

(iv) The set of points of discontinuity of the function $f : [0, 1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}.$$

(c) If $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are bounded functions such that $fg : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, then

(i) both f and g are Riemann integrable on $[a, b]$.

(ii) at least one of f and g is Riemann integrable on $[a, b]$.

(iii) If f is Riemann integrable on $[a, b]$, then g is Riemann integrable on $[a, b]$.

(iv) f and g may not be Riemann integrable on $[a, b]$.

(d) $f:[-1,1]\to\mathbb{R}$ is defined by $f(x)=\begin{cases}0, & -1\le x<0\\ 1, & 0\le x\le 1\end{cases}$. Then

(i) f is Riemann integrable and has primitive on $[-1, 1]$.

(ii) f is Riemann integrable but does not have primitive on $[-1, 1]$.

(iii) f is not Riemann integrable but has primitive on $[-1, 1]$

(iv) f is not Riemann integrable and also does not have primitive on $[-1, 1]$.

(e) $\int_1^{\infty}\frac{x^{n-1}}{x+1}dx$ converges for

(i) $n>1$ (ii) $n<1$

(iii) $n>0$ (iv) $n<0$.

(f) $\int_{-\infty}^{\infty}e^{-x^2}dx$ is equal to

(i) $\frac{\sqrt{\pi}}{2}$ (ii) $\sqrt{\pi}$

(iii) $2\sqrt{\pi}$ (iv) $\sqrt{\frac{\pi}{2}}$.

(g) Let $f_n(x)=x^n$ for each $n\in\mathbb{N}$. Then $\{f_n\}_{n=1}^{\infty}$ is uniformly convergent on

(i) $(0, 1)$ (ii) $[0, 1]$

(iii) $\left(\frac{3}{5}, 1\right]$ (iv) $\left[0, \frac{3}{5}\right)$

(h) $\sum_{n=1}^{\infty}\frac{1}{n^2+n^3x^4}$ is

(i) uniformly convergent but not absolutely convergent on $\mathbb{R}$.

(ii) uniformly convergent and absolutely convergent on $\mathbb{R}$.

(iii) absolutely convergent but not uniformly convergent on $\mathbb{R}$.

(iv) neither absolutely convergent nor uniformly convergent on $\mathbb{R}$.

(i) The radius of convergence of $\sum_{n=1}^{\infty} \frac{(2n)!x^n}{(n!)^2}$ is

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) $\frac{1}{4}$ (iv) $\frac{2}{3}$.

(j) If $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is the Fourier series of the function $x\sin x$ in $[-\pi, \pi]$, then identify the incorrect statement.

(i) $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges to $x\sin x$ in $[-\pi, \pi]$

(ii) $\sum_{n} a_n$ is convergent

(iii) $\sum_{n} n^2 b_n$ is convergent

(iv) $a_n \neq 0$ for every $n \in \mathbb{N}$.

2. Answer ***any three*** questions :

(a) Define primitive of a function on $[a, b]$. Prove that every continuous function on a closed and bounded interval has a primitive there. 1+4

(b) (i) Define 'negligible set' in $\mathbb{R}$. State the characterization theorem for Riemann integrability in terms of negligible sets.

(ii) Discuss the Riemann integrability of the function $f: [-500, 500] \to \mathbb{R}$ defined by $f(x) = 2 - [x]$. ($[x]$ denotes the largest integer not exceeding x) (1+2)+2

(c) (i) Let $f: [0, 1] \to \mathbb{R}$ be defined by $f(x) = \lim_{n\to\infty} (\sin 2x)^n$. Check whether f is Riemann integrable on $[0, 1]$.

(ii) Let f, g be Riemann integrable on $[a, b]$ and $\int_a^b f^2 = 0$. Prove or disprove : $\int_a^b fg = 0$. 2+3

(d) (i) If $f: \mathbb{R} \to \mathbb{R}$ is Riemann integrable on $[0, 1]$ and $f(x+y) = f(x) + f(y)\ \forall x \in \mathbb{R}$, then show that $\int_0^1 f = \frac{f(2022)}{4044}$.

(ii) Prove or disprove : $|f|$ is Riemann integrable on $[a, b]$ implies f is Riemann integrable on $[a, b]$. 3+2

(e) (i) State Fundamental Theorem of integral calculus.

(ii) Let $f: [0, 3] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & 0 \le x \le 1 \\ 1, & 1 < x \le 2 \\ 2, & 2 < x \le 3 \end{cases}.$$

Let $F(x) = \int_0^x f(t)dt,\ x \in [0, 3]$. Find the function F. Examine whether F is continuous on $[0, 3]$. 1+(2+2)

3. Answer ***any two*** questions :

(a) Discuss the convergence of the improper integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$. 5

(b) (i) Show that $\int_1^\infty \frac{\sin x dx}{x^p}$ converges absolutely for $p > 1$ and conditionally for $0 < p \le 1$.

(ii) Show that $\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, k^2 < 1$ is convergent. 3+2

(c) State Abel's Test and use it to test the convergence of $\int_0^\infty e^{-ax}\frac{\sin x}{x}dx, a \ge 0$. 1+4

(d) (i) Show that $\Gamma(n+1) = n!$ for any $n \in \mathbb{N}$.

(ii) Examine the convergence of $\int_0^\infty \frac{\sin x + 2}{\log x}dx$. 2+3

4. Answer ***any four*** questions :

(a) Prove that a sequence $\{f_n\}_{n=1}^{\infty}$ of functions $f_n : [a, b] \to \mathbb{R}$ converges uniformly to some function $f : [a, b] \to \mathbb{R}$ if and only if $\lim\limits_{n\to\infty} M_n = 0$ where $M_n = \sup\limits_{x\in[a,b]} |f_n(x) - f(x)|$, for each $n \in \mathbb{N}$. 5

(b) Examine the uniform convergence of the sequence $\{g_n\}_{n=1}^{\infty}$ of functions, where for each $n \in \mathbb{N}$,

$g_n(x) = nxe^{-nx^2}$, $x \in [0,1]$ 5

(c) Discuss the uniform convergence of $\sum\limits_{n=1}^{\infty} n^2x^2e^{-n^2|x|}$ on $\mathbb{R}$. 5

(d) Let $D \subseteq \mathbb{R}$ and for each $n \in \mathbb{N}$, let $f_n : D \to \mathbb{R}$ be continuous on D. Then prove that the uniform sum function of $\sum\limits_{n} f_n$ is continuous on D. Is uniformity a necessary condition for the conclusion to hold? Justify your answer. 3+2

(e) (i) Find the radius of convergence of the power series $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n!}{n^n} x^n$.

(ii) Prove or disprove : Radius of convergence of a power series remains invariant under term-by-term differentiation. 3+2

(f) State Abel's theorem for uniform convergence of power series. Use it to prove that the sum of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is $\log 2$. 1+4

(g) Obtain the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$ defined by $f(x) = \begin{cases} -\cos x, & -\pi \le x < 0 \\ \cos x, & 0 \le x \le \pi \end{cases}$.

Hence, find the sum of the series $\frac{2}{1\cdot 3} - \frac{6}{5\cdot 7} + \frac{10}{9\cdot 11} - \ldots$ 3+2