2022

MATHEMATICS — HONOURS

Paper : CC-9

(Partial Differential Equation and Multivariate Calculus-II)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

All symbols have their usual meaning.

Group – A

(Marks : 20)

1. Answer all questions with proper justification (one mark for correct answer and one mark for justification) : $(1+1)\times 10$

(a) Nature of the partial differential equation (PDE) $u_{xx}^2 + u_x^2 + \sin u = e^y$ is

(i) non-linear first order (ii) non-linear second order

(iii) linear first order (iv) none of these.

(b) Elimination of the arbitrary constants a and b from the equation $\log_e(az - 1) = x + ay + b$ gives the PDE

(i) $\left(1 + \frac{\partial z}{\partial x}\right)\frac{\partial z}{\partial y} = z\frac{\partial z}{\partial x}$ (ii) $\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x}$

(iii) $\left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}$ (iv) none of these.

(c) Characteristic curves of the PDE $u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0$ is given by

(i) $y + (1 - 2i)x = c_1,\ y + (1 + 2i)x = c_2$ (ii) $y - (1 - 2i)x = c_1,\ y - (1 + 2i)x = c_2$

(iii) $y - (1 + 2i)x = c_1,\ y - (1 + 2i)x = c_2$ (iv) none of these.

(d) $u_{xx} - \sqrt{x}u_{xy} + x\,u_{yy} = e^{\frac{x}{2}}$ for all $x \geqslant 0$ is

(i) hyperbolic for all values of x. (ii) parabolic for all values of x.

(iii) elliptic for all values of x. (iv) parabolic for $x = 0$ and elliptic for $x > 0$.

(e) $\left(x^2 - y^2 - z^2\right)\frac{\partial z}{\partial x} + 2xy\frac{\partial z}{\partial y} = 2xz$ has a solution

(i) $x^2 + y^2 + z^2 = z f\left(\frac{y}{z}\right)$ (ii) $x^2 - y^2 - z^2 = y f\left(\frac{y}{z}\right)$

(iii) $x^2 + y^2 + z^2 = f\left(\frac{y}{z}\right)$ (iv) $x^2 - y^2 - z^2 = z f\left(\frac{y}{z}\right)$.

(f) The complete solution of the non-linear partial differential equation $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = c^2$ is

(i) a cone (ii) a cylinder

(iii) a sphere (iv) none of these.

(g) Value of $\iint xy\, dx\, dy$ over the region bounded by $xy = 1, y = 0, y = x, x = 1$ is

(i) $1/8$ (ii) $1/4$

(iii) 1 (iv) $1/2$

(h) If the order of integration $\int_0^1 dy \int_{x=y}^{x=\sqrt{y}} f(x, y)\, dx$ is interchanged, then it will take the form

(i) $\int_0^1 dx \int_{y=x^2}^{y=x} f(x, y)\, dx$ (ii) $\int_0^1 dx \int_x^1 f(x, y)\, dx$

(iii) $\int_0^1 dx \int_{x^2}^1 f(x, y)\, dx$ (iv) none of these.

(i) If $\vec{F} = x^2 y\hat{i} + xz\hat{j} + 2yz\hat{k}$, then the value of $\operatorname{div}\{\operatorname{curl} \vec{F}\}$ is

(i) 1 (ii) 0 (iii) 2 (iv) $\hat{i} + \hat{k}$.

(j) The work done by a particle in the force field $\vec{F} = 3x^2\hat{i} + (2xz - y)\hat{j} + z\hat{k}$ along straight line from (0, 0, 0) to (2, 1, 3) is

(i) 16 units (ii) 22 units

(iii) 14 units (iv) 42 units.

Group – B

(Marks : 21)

Answer ***any three*** questions.

2. (a) Apply Charpit's method to find the complete integral of the PDE $(p+q)(px+qy)=1$.

   (b) Form a PDE by eleminating the arbitrary function φ and ψ from the relation $u(x, y) = y\varphi(x) + x\psi(y)$. 4+3

3. Using method of separation of variables solve the PDE $4z_x + z_y = 3z$ under the condition $z = 3e^{-y} - e^{-5y}$ at $x = 0$. 7

4. Using $\eta = x + y$ as one of the transformation variable, obtain the canonical form of

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0$$

   and hence solve it. 7

5. A tightly stretched string of length l with fixed end points is initially at rest in its equilibrium position, and each of its points is given a velocity v, which is given by

$$v(x) = \begin{cases} cx, & 0 \le x < l/2 \\ c(l-x), & \frac{l}{2} \le x \le l \end{cases}$$

   Find the displacement, c being the wave speed. 7

6. Solve the following initial boundary value problem

$$u_t = u_{xx}\ (0 < x < \lambda,\ t > 0)$$

   subject to the conditions $u(x, 0) = 3\sin n\pi x$ (n a+ve integer)

$$u(0, t) = u(\lambda, t) = 0.$$

   7

Group – C

(Marks : 24)

Answer ***any four*** questions.

7. Using differentiation under the sign of integration find the value of $\int_0^{\infty} e^{-a^2x^2} \cos^2 bx\, dx$. 6

8. Evaluate the integral $\iint \frac{dx\,dy}{\left(1+x^2+y^2\right)^2}$ taken over the triangle with vertices at (0, 0), (2, 0) and $\left(1, \sqrt{3}\right)$. 6

9. Find the value of the integral $\iiint\limits_E \frac{dx\,dy\,dz}{x^2 + y^2 + (z-2)^2}$, where $E = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. 6

10. Define conservative vector field $\vec{F}$ and express its relation with the scalar potential $\varphi(x, y, z)$. Write down Gauss's divergence theorem in case of an irrotational vector over the hemisphere centered at origin. 4+2

11. Find $\oint\limits_c x\,dy + y\,dx$ bounded by the closed contour of astroid with $x = a\cos^3 t$ and $y = a\sin^3 t$. 6

12. Find the surface area of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$. 6

13. Prove that the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$ is

$\frac{2}{9}(3\pi - 4)a^3$. 6

———