2022

MATHEMATICS — HONOURS

Paper : DSE-A-1.1

(Advanced Algebra)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Notations have usual meanings.

Group - A

(Marks : 20)

1. Answer ***all*** questions. In each question **one** mark is reserved for selecting the correct option and **one** mark is reserved for justification : (1+1)×10

(a) Let S be a G-set where G is a group and S is a non-empty set. Then the relation ρ on S defined by : for $a, b \in S$, $a \rho b$ if and only if $ga = b$ for some $g \in G$ is

(i) reflexive and symmetric but not transitive

(ii) an equivalence relation

(iii) reflexive and transitive but not symmetric

(iv) symmetric and transitive but not reflexive.

(b) A group of order 20 has

(i) 2 Sylow 5-subgroups (ii) 3 Sylow 5-subgroups

(iii) 1 Sylow 5-subgroup (iv) 4 Sylow 5-subgroups.

(c) Suppose that G is a finite group which has only two conjugacy classes. Then G has

(i) one element (ii) two elements

(iii) three elements (iv) four elements.

(d) The ring of integers $(Z, +, \cdot)$ is

(i) a regular ring (ii) not an integral domain

(iii) not a regular ring (iv) a field.

(e) The units of $(Z_{10}, +, \cdot)$ are

(i) [1], [3], [7], [9] (ii) [1], [4], [5], [9]

(iii) [1], [7], [9] (iv) [1], [9].

(f) The polynomial $x^4 + x^3 + x^2 + x + 1$ is

(i) reducible in $Z[x]$

(ii) reducible in $Q[x]$

(iii) irreducible in $Z[x]$ but reducible in $Q[x]$

(iv) irreducible in both $Z[x]$ and $Q[x]$.

(g) Let S be a finite G-set, where $|G|$ is 81. Then if $H = \{a \in S : ga = a, \forall g \in G\}$, then the difference of orders of S and H is divisible by

(i) 81 (ii) 27

(iii) 9 (iv) 3.

(h) Which one of the following is correct?

(i) The polynomial ring over the ring of real numbers is a PID.

(ii) The polynomial ring over the ring of integers is a PIR.

(iii) The ring of integers is not a PIR.

(iv) The polynomial ring over the ring of integers is not a PID.

(i) The quotient ring $Z/\langle n \rangle$ is a field, if

(i) n is an integer (ii) n is a natural number

(iii) n is a prime number (iv) none of these.

(j) If R is a regular ring and A is a right ideal and B is a left ideal of R, then

(i) $A \cap B = AB$ (ii) $A \cap B \neq AB$

(iii) $A \cup B = AB$ (iv) $A \cup B \neq AB$.

Group - B

(Marks : 15)

2. Answer ***any three*** questions :

(a) Let G be a finite group of order n and let p be a prime number where p divides n. Prove that G has a subgroup of order p. 5

(b) Show that every group of order 45 has a normal subgroup of order 9. 5

(c) Write all Sylow 2-subgroups of S_3 with proper justification. 5

(d) (i) Let G be a simple group of order 168. Show that G has 8 Sylow 7-subgroups.

(ii) Let G be a group and S be a G-set. Define stabilizer of an element $a \in S$. 4+1

(e) (i) Prove that no group of order pq is simple, where p, q are prime numbers.

(ii) Let G be a group of order 35. Prove that G is commutative. 3+2

Group - C

(Marks : 30)

3. Answer ***any six*** questions :

(a) (i) Prove that the ring Z of all integers is a principal ideal domain.

(ii) Give an example of a ring which is not a principal ideal domain. Justify your answer. 3+2

(b) (i) Define a Euclidean domain. Prove that every Euclidean domain is a principal ideal domain.

(ii) Give an example of a principal ideal domain which is not a Euclidean domain (justification is not required). 1+3+1

(c) Prove that every integral domain can be embedded in a field. 5

(d) Define a unique factorization domain. Prove that in a unique factorization domain, every irreducible element is prime. 1+4

(e) When is a ring said to be regular? Prove that every field is a regular ring. Is every integral domain a regular ring? Justify your answer. 1+2+2

(f) Show that $-1+2i$ is a g.c.d of $11+3i$ and $1+8i$ in $Z[i]$. 5

(g) Show that ℓ.c.m. of 2 and $1+\sqrt{-5}$ does not exist in $Z[\sqrt{-5}]$. 5

(h) (i) When a ring is said to satisfy ascending chain condition for principal ideals (ACCP)?

(ii) Prove that every PID satisfies ACCP. 1+4

(i) Let $f(x) \in F(x)$, where F is a field, be a polynomial of degree 2 or 3. Then prove that $f(x)$ is irreducible over F if and only if $f(x)$ has no roots in F. 5

(j) (i) If $f(x) = x^4 + 2x^2 + 1$, prove that it has no root in Q but is reducible over Z.

(ii) Prove that $f(x) = x^5 + 15x^3 + 10x + 5$ is irreducible in $Z[x]$. 3+2
