2022

PHYSICS — HONOURS

Paper : DSE-A-1(a) and DSE-A-1(b)

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Paper : DSE-A-1(a)

(Advanced Mathematical Methods)

Full Marks : 65

Answer ***question nos.*** **1** and **2**, and ***any four*** questions from the rest (***Q.*** **3** to ***Q.*** 8).

1. Answer ***any five*** questions : 2×5

(a) Prove that any set of vectors that include the null vector is linearly dependent.

(b) Is it possible to write $v = (1, -2)$ in $\mathbb{R}^2$ as a linear combination of the vectors $u_1 = (1, -3)$ and $u_2 = (2, -4)$?

(c) What is a 'functional'? Show that for any vector $|a\rangle$ of a complex vector space $V(C)$, $f(|v\rangle) = \langle a|v\rangle$ defines a linear functional for $|v\rangle \in V(c)$.

(d) How do the contravariant and the covariant components of a vector transform under a general co-ordinate transformation?

(e) The moment of inertia tensor of a square lamina, having mass 'M' and sides of length 'L' is given by :

$$\begin{pmatrix} ML^2/3 & -ML^2/4 & 0 \\ -ML^2/4 & ML^2/3 & 0 \\ 0 & 0 & 2ML^2/3 \end{pmatrix}$$, where its two adjacent sides are taken

as the X– and Y– axis. Find its moment of inertia about a diagonal.

(f) Show that in an Abelian group, every element is a (conjugacy) class by itself.

(g) All the elements of a group G_1 is mapped to a single element of another group G_2. Can it be a homomorphism?

2. Answer ***any three*** questions :

(a) Prove that : $|\langle a|b\rangle| \leq \|a\| \cdot \|b\|$ for any two complex vectors $|a\rangle, |b\rangle$, where $\|a\|$ and $\|b\|$ stands for their norms. 5

(b) (i) Express the scalar triple product of three vectors, using the Levi-Civita symbol and hence prove that : $\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\vec{B}\cdot\left(\vec{C}\times\vec{A}\right)$.

(ii) From the equation $\partial_\mu F^{\mu\nu}=J^\nu$, where $F^{\mu\nu}$ is the electromagnetic field strength tensor, and J^ν is the 4-current density, obtain the equation of continuity in the three-vector form. 2+3

(c) Prove that, if A^μ are the contravariant components of a 4-vector, then $A_\mu=g_{\mu\nu}A^\nu$ transform in the covariant fashion, where $g_{\mu\nu}$ is the 4-dimensional metric tensor. How do you define $g^{\mu\nu}$? 4+1

(d) Let (G, o) be a group and $H\subseteq G$. Prove that, if $h_1, h_2\in H \Rightarrow (h_1\bullet h_2^{-1})\in H$, then H is a sub group of G. 5

(e) Consider a two-object permutation group. Construct its group multiplication table and show that this is an abelian group. Construct a valid matrix representation of the elements of the group. 2+1+2

3. (a) Check whether the set of matrices

$$\begin{pmatrix}1&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\text{and}\begin{pmatrix}0&1\\1&0\end{pmatrix}$$

is linearly independent.

(b) A linear operator $\mathbb{R}^3\to\mathbb{R}^3$ takes $\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$ to $(x+2y+z)\hat{i}+(x-y+2z)\hat{j}+(3x-y+z)\hat{k}$.

Find the matrix representation of the operator.

(c) Let V be the vector space over all 2×2 matrices over the real field R. Show that W is not a subspace of V, where

(i) W consists of all matrices with zero determinant.

(ii) W consists of all matrices for which $A^2=A$. 3+3+(2+2)

4. (a) Define an 'inner product space'.

(b) Show that a set of orthogonal non-null vectors (i.e., each pair of distinct vectors in the set is orthogonal) is always linearly independent.

(c) Inner product is defined over a vector space of functions as : $\langle f(x)\,|\,g(x)\rangle=\int_{-\infty}^{\infty}e^{-x^2}f(x)\,g(x)dx$.

Use Gram-Schmidt procedure to orthogonalize the set of functions : $\{1, x, x^2\}$ with respect to the inner product defined above. (The resultant functions need not be normalized.) 3+2+5

5. (a) Derive an expression for the moment of inertia tensor of a general body.

(b) Three particles, each of mass 'm' are situated at the points : (1,1,1), (–1,1,2) and (0,–1,–2), Construct the moment of inertia tensor for the system. 5+5

6. (a) What is a 'four vector'? Write the contravariant components (p^μ) and the covariant components (p_μ) of the momentum 4-vector and evaluate the 4-scalar p^2. (take the four-dimensional metric tensor $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$).

(b) How are the components of the electric and the magnetic field related to those of the electromagnetic field strength tensor $F^{\mu\nu}$?

Show that the relation : $\partial_\mu F^{\mu\nu} = J^\nu$ produces the inhomogeneous Maxwell's equations, while expressed in 3-vector notation. (1+1+1+1)+(2+4)

7. (a) Show that the same element cannot occur twice in a column of a group multiplication table.

(b) The multiplication table of a group is given below :

	E	R	R^2	σ_A	σ_B	σ_C
E	E	R	R^2	σ_A	σ_B	σ_C
R	R	R^2	E	σ_C	σ_A	σ_B
R^2	R^2	E	R	σ_B	σ_C	σ_A
σ_A	σ_A	σ_B	σ_C	E	R	R^2
σ_B	σ_B	σ_C	σ_A	R^2	E	R
σ_C	σ_C	σ_A	σ_B	R	R^2	E

(i) Find the inverses of the elements R and σ_A.

(ii) Find the conjugacy classes of R and σ_A. 2+(2+6)

8. (a) Consider a 2-dimensional anticlockwise rotation by an angle θ that takes the orthogonal (x, y) axes to (x', y').

(i) How many parameters are needed to parametrise this group? Is it an abelian group?

(ii) Find the matrix representation X of the generator of this group.

(iii) Show that $e^{i\theta x} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

(b) Consider the algebra of a Lie group

$$[X_a, X_b] = i f_{abc} X_c$$

where f_{abc} s are all real. Show that

(i) $f_{abc} = -f_{bac}$ for all values of a and b.

(ii) Show that $-X_a^*$ matrices also satisfy the same Lie algebra. (2+2+2)+(2+2)

Paper : DSE-A-1(b)

(Laser and Fibre Optics)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Answer ***question nos.*** **1** and **2** and ***any four*** questions from the rest.

1. Answer ***any five*** questions : 2×5

   (a) Calculate the numerical aperture and acceptance angle of an optical fibre for core and cladding refractive indices 1.51 and 1.48 respectively.

   (b) What do you mean by monochromaticity of an optical beam?

   (c) Find the intensity of a laser beam of 15 mW power and having a diameter of 1.3 mm. Assume the intensity to be uniform across the beam.

   (d) Why a three-level laser normally provides pulsed output?

   (e) Distinguish between step index fibre and graded index fibre structure.

   (f) What is Pockels effect?

   (g) What is meant by self-focussing?

2. Answer ***any three*** questions :

   (a) A step index fibre is made with a core of index 1.52, a diameter of 30 μm and a fractional difference of index 0.0007. It is operated at a wavelength 1.3 μm. Find

      (i) The V-number of fibre and

      (ii) The approximate number of modes the fibre can carry. 3+2

   (b) Explain the principle of holography with diagram. 5

   (c) A two-level laser has n_1 and n_2 number of atoms/unit volume in level 1 and level 2 respectively. Then show that

   $$\Delta n = \frac{n_o}{1+2P\tau}, \text{ where } n_o = n_1 + n_2, \quad \Delta n = n_1 - n_2$$

   τ is the overall lifetime of upper level and P is pumping rate. 5

   (d) Show that (i) confocal resonator and (ii) planer resonator can act as stable optical resonator. Write down the necessary equation. Hence, draw the stability diagram and indicate the positions of the above mentioned resonators in the diagram. 1+1+1+1+1

(e) (i) With proper schematic diagram discuss how the Fabry Perot etalon can help in mode selection in an optical resonator.

(ii) The cavity of He-Ne laser emitting at 632.8 nm consists of two mirrors separating a distance of 35 nm. If the oscillation in the laser cavity occurs at frequency within the gain bandwidth of 1.3 GHz, then find out the number of longitudinal modes allowed in the cavity. 3+2

3. (a) Find an expression for the intensity of Second Harmonic Generation at the exit surface of a material.

(b) From the expression for the intensity obtain the criterion for phase matching.

(c) Why is it called refractive index criterion? 6+2+2

4. (a) Deduce the relation, with a suitable diagram, between the Einstein's A and B coefficients.

(b) At thermal equilibrium, obtain the ratio of the number of spontaneous to stimulated emissions.

(c) A signal of 100 mW is injected into a fibre. The outcoming signal from the other end is 40 mw. Find the loss in decibel (dB). 6+2+2

5. (a) Let a step index single mode fibre is characterised by

$$\begin{aligned} n(r) &= n_1 \quad \text{for } 0 < r \le a \text{ (core)} \\ &= n_2 \quad \text{for } r > a \text{ (cladding)} \end{aligned}$$

where $n(r)$ is refractive index, k_0 = free space wave number, β = the wave propagation constant and a = radius of the core. Show that the guided mode is possible when $n_2^2 < \frac{\beta^2}{k_0^2} < n_1^2$.

(b) In the case of multimode graded index fibre, using power law profile, show that a parabolic index fibre can accommodate nearly 25 modes.

(c) A step index fibre with $n_{core} = 1.485$ and $n_{cladding} = 1.455$ has a core radius, $a = 5.92\times10^{-6}$ m. Calculate the operational wavelength (λ_0) for which waveguide parameter, $V = 10$. 6+2+2

6. (a) Define quality factor in a resonator cavity.

(b) If W_0 represents the energy stored in a mode at $t = 0$, then find out the expression for the energy stored in that mode at time t in terms of quality factor.

(c) Hence find out the passive cavity lifetime.

(d) If the resonator has two end mirrors M_1 and M_2 placed at a distance d from each other with reflectivity R_1 and R_2, write down the expression for energy stored in the mode which corresponds to a pair of reflection. (α_C is the net power absorption coefficient and n_0 is the refractive index of the active medium).

(e) Find out the quality factor and passive cavity lifetime in terms of R_1, R_2, n_0, d and α_C. 2+2+1+1+2+2

7. (a) Write down the allowed frequencies of the optical modes in a rectangular cavity of dimension $2a \times 2b \times d$. Hence obtain the expression for the frequencies in case of an open resonator with proper approximation.

(b) Show that the difference of frequency between two longitudinal mode is $C/2d$ for $a = b$.

(c) Find out the frequency difference between two consecutive transverse mode for a single longitudinal mode. 2+3+2+3

8. (a) Write down the rate equations for a 3-level laser explaining the symbols.

(b) Obtain the condition for population inversion from the above equations.

(c) For a Ruby laser, number of atoms per cm^3 is 1.6×10^{19}, spontaneous lifetime is 3×10^{-10}s and average pump frequency is 6.25×10^{14} Hz. Calculate the threshold pump power. 3+5+2