2022

STATISTICS — HONOURS

Paper : CC-3

(Mathematical Analysis)

(Unit - 1 to 4)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer ***any ten*** questions. If you answer more than ten, then only the first ten attempted will be checked. 2×10

(a) Let x and y be two real numbers such that $0 \leq x \leq y$. Furthermore if it is given that $0 \leq y < \epsilon$ for every $\epsilon > 0$, show that $x = y = 0$.

(b) Find all the rational numbers x that satisfy the inequality $|x-2| \leq x+1$. Write your solution in the form of a set.

(c) Find the supremum of the set $A = \{x \in \mathbb{R} \mid x > 2/x, x \neq 0\}$.

(d) Give examples of the following :

(i) A convergent sequence with a monotonically increasing, and a monotonically decreasing, subsequences.

(ii) A divergent sequence having a convergent, and another divergent, subsequences.

(e) Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$. Provide examples of the following situations :

(i) $\{a_n\}$ is bounded and $b_n > 0 \ \forall n \geq 1$.

(ii) $\{b_n\}$ diverges to infinity and $a_n > 0 \ \forall n \geq 1$.

(f) Let $\{a_n\}$ be a sequence of real numbers such that $a_{n+1} = \sqrt{a_n + 1}$, $n \geq 1$ with $a_1 = 1$. Check whether the sequence is convergent.

(g) Let $\{a_n\}$ be a sequence of real numbers and $M \geq 1$ be any integer. Show that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series $\sum_{n=1}^{\infty} a_{M+n}$ converges.

(h) Does the series $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ converge? Justify.

(i) Suppose both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent. Show that the series $\sum_{n=1}^{\infty} a_n b_n$ is also absolutely convergent.

(j) Prove that $\lim_{x\to 0} \cos(1/x)$ does not exist but that $\lim_{x\to 0} x\cos(1/x) = 0$.

(k) Write the sequential definition of limit of a function.

(l) Consider a function :

$$f(x) = \begin{cases} x^2 - 6x, & \text{if } x \in (0,6) \\ 10, & \text{if } x = 7 \end{cases}$$

Check the continuity of the function at $x = 7$.

(m) Using Taylor's Theorem show that $e^{\pi} > \pi^{e}$.

(n) Discuss the use of polar transformation in double integration with an example.

(o) Consider the function $f(x, y) = y^2 - x^2$, $(x, y) \in \mathbb{R}^2$. Does there exist any extremum of the function?

2. Answer ***any three*** questions :

(a) Define maximum and minimum of a set. Show that the set $A = (0, 1)$ does not have a maximum. 2+3

(b) State and prove Monotone Convergence Theorem for a sequence of real numbers. 5

(c) 'Root test is more powerful than ratio test'. — Justify the statement with an example. 5

(d) Define continuity and uniform continuity of a function. If a function $f : [a, b] \to \mathbb{R}$ is continuous then show that it is bounded on $[a, b]$. 2+3

(e) Evaluate :

$$\int_{\mathcal{D}} xy \, dx \, dy$$

where $\mathcal{D} = \{(x, y) : ax^2 + by^2 + 2hxy \le c^2\}$ with a, b, h and c real numbers such that $a > 0$, $c > 0$ and $(ab - h^2) > 0$. 5

3. Answer ***any three*** questions :

(a) (i) Prove that between any two real numbers there are infinitely many irrational numbers.

(ii) Show that $\bigcap_{n=1}^{\infty}\left(0,\frac{1}{n}\right)=\phi(\text{null set})$. 5+5

(b) (i) Define absolute convergence and conditional convergence of infinite series. Provide suitable examples.

(ii) Show that a series $\sum_{n=1}^{\infty}a_n$ of real numbers is convergent absolutely if and only if $\sum_{n=1}^{\infty}a_n^+$ and $\sum_{n=1}^{\infty}a_n^-$ are both convergent, where $a_n^+=\max(a_n,0)$ and $a_n^-=-\min(a_n,0)$. (2+2)+6

(c) (i) State Lagrange's Mean Value Theorem and discuss one of its applications.

(ii) Give an example of a function which is uniformly continuous on [0,1], differentiable on (0,1) but the derivative is not bounded on (0,1). Justify your answer. (2+2)+6

(d) (i) Discuss the use of Lagrange multipliers to find extremum of a function of several variables satisfying some constraints.

(ii) Let $f:\mathbb{R}^2\to\mathbb{R}$ be the function defined by $f(x,y)=x^2-12y$. Find the maximum and minimum values of the function f on the circle $x^2+y^2=49$. 5+5

(e) (i) Stating necessary result(s), find the value of the following limit : $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

(ii) Stating necessary result(s), evaluate : $\int_0^{\frac{\pi}{2}}\frac{\sin(x)}{x^{3/2}}dx$. 5+5